Solving the nonlinear complementarity problem

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Abstract

In order to solve the nonlinear complementarity problem we propose a simplicial restart algorithm that subdivides the set on which the problem is defined into simplices and generates from an arbitrarily chosen starting point a piecewise linear either leading to an approximate solution or diverging towards infinity. We will give a convergence condition under which the algorithm will find an approximate solution. If the accuracy of the approximate solution is not sufficient the algorithm can be restarted at the approximate solution with a finer simplicial subdivision. The piecewise linear path generated by the algorithm is followed by a sequence of adjacent simplices of varying dimension.

KEYWORDS: Nonlinear Complementarity Problem, simplicial variable dimension restart algorithm, path following algorithm, piecewise linear path

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1 Introduction

One of the most well-known problems in the field of mathematical programming is the so-called nonlinear complementarity problem. It's also a problem that is frequently met when solving systems of nonlinear equations, or computing economic equilibria and fixed points. The nonlinear complementarity problem (NLCP) is defined as follows.

Given a continuous function f from $\mathbf{R}^n$ to $\mathbf{R}^n$, find an $x^* \in \mathbf{R}^n$ such that

$$x^* \geq 0, \quad f(x^*) \geq 0, \quad x^{*\top} f(x^*) = 0. \tag{1}$$

The NLCP is a special case of the NLCP with lower and upper bounds which can be seen by taking the zero vector as a lower bound and letting the upper bound go to infinity. A description of the NLCP with lower and upper bounds can be found in Kremers and Talman (1990) together with a description of an algorithm solving the NLCP with finite lower and upper bounds.

In this paper we will describe an algorithm solving the NLCP defined in (1). This algorithm will be a generalisation of the algorithm developed in Kremers and Talman (1990). Our algorithm is a path-following algorithm starting in an arbitrarily chosen point $v \in \mathbf{R}^n_+$. The description of the path to be followed by the algorithm is given in Section 2 while Section 3 describes the algorithm itself. Also a convergence condition for the algorithm will be given because the unbounded region in (1) gives rise to possible divergence of the algorithm. The convergence condition guarantees the existence of an upper bound to the points generated by the algorithm. Therefore, we cannot apply the algorithm developed in Kremers and Talman (1990) in order to solve (1). In Section 4 we describe an appropriate simplicial subdivision of $\mathbf{R}^n_+$.

2 The path to be approximated by the algorithm

Starting in an arbitrary chosen point $v \in \mathbf{R}^n_+$ the algorithm follows approximately a path of points $x \in \mathbf{R}^n_+$ such that x satisfies

$$\begin{array}{rcccll} \max\{0,(1-\rho)v_i\} & = & x_i & & & \text{if } f_i(x) > 0 \\ \max\{0,(1-\rho)v_i\} & \leq & x_i & \leq & v_i + \rho & \text{if } f_i(x) = 0 \\ & & x_i & = & v_i + \rho & \text{if } f_i(x) < 0 \end{array} \tag{2}$$

for some $\rho \geq 0$. Under some regularity and non-degeneracy conditions the set of points x satisfying the conditions in (2), for $\rho \geq 0$, form piecewise smooth curves. Each of these curves is either a loop or a path. One of these paths, say P, has v as an end point for $\rho = 0$. If P has another end point, say x^*, the x^* is a solution to (1). Otherwise the path P will go towards infinity and no solution to (1) will be found. The algorithm follows approximately this path P by making f linear on each simplex of a simplicial subdivision of $\mathbf{R}^n_+$. For an illustration of the set $\mathcal{H}(t) \cap \mathbf{R}^n_+$ for different values of t we refer to Figure 1. Without loss of

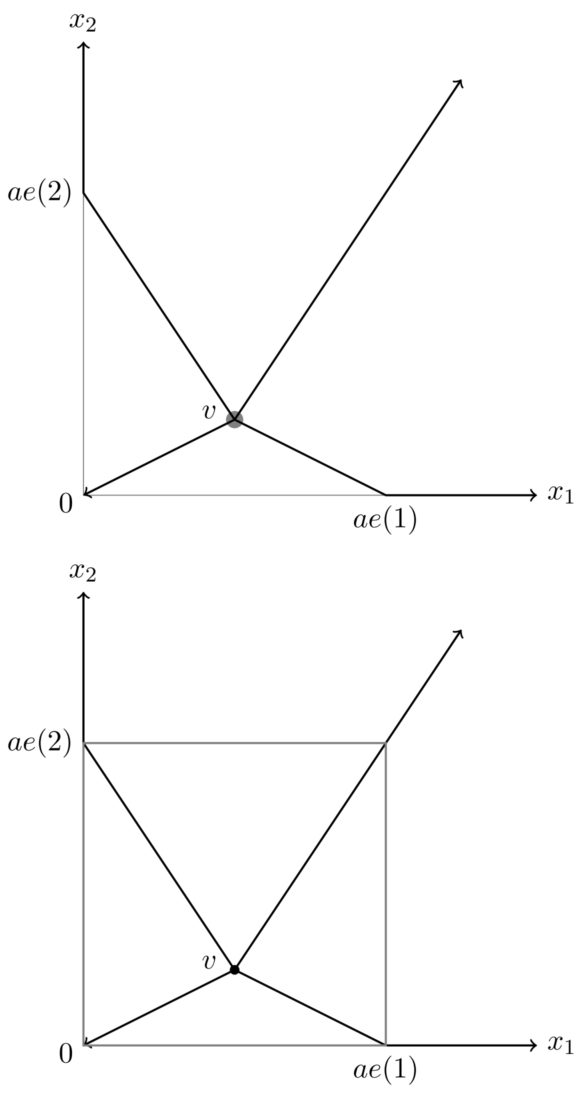

Figure 1: The subset bd $\left(\mathcal{H}(\rho) \cap \mathbf{R}^2_+\right)$ for $\rho = 0, \frac{1}{2},\ 1$, resp. $\rho > 1$, given $v = (2,1)^\top$ and $a = 4$.

generality we assume that no component of $f(v)$ equals zero. Then, through increasing ρ from zero the path P leaves v by increasing x_i from v_i such that $x_i = v_i + \rho$ if $f_i(v) < 0$ and by decreasing x_i from v_i such that

$x_i = \max\{0, (1-\rho)v_i\}$ if $f_i(v) > 0$, for all $i \in \{1, \ldots, n\}$. If along the path P, at a point x satisfying (2), $f_j(x)$ becomes zero for some $j \in \{1, \ldots, n\}$ while $x_j = v_j + \rho$ (or $x_j = \max\{0, (1-\rho)v_j\}$) then either x solves (1) or the path continues by decreasing (increasing) x_j from $v_j + \rho$ ($\max\{0, (1-\rho)v_j\}$) and keeping $f_j(x) = 0$. If at a point x on P, satisfying (2), x_j becomes equal to $v_j + \rho$ (or $\max\{0, (1-\rho)v_j\}$), for some $j \in \{i \mid f_i(x) = 0\}$, then the path P continues by decreasing (increasing) $f_j(x)$ from 0 and keeping x_j equal to $v_j + \rho$ (or $\max\{0, (1-\rho)v_j\}$). Finally, if at a point x on P, ρ becomes equal to 1 and $f(x) \geq 0$, then $f_i(x) = 0$ when $0 < x_i \leq v_i + 1$ and $f_i(x) \geq 0$ when $x_i = 0$, so x is a solution of (1). Otherwise, ρ will be increased further, keeping $x_i = 0$ for all i such that $f_i(x) > 0$.

3 The algorithm

The algorithm approximately follows the path P described in Section 2 by generating a piecewise linear (piecewise linear) path $\overline{P}$ connecting v with an approximate solution $\overline{x}$ of (1) or diverging towards infinity. For a description of this piecewise linear path we approximate the function f by a piecewise linear approximation F.

To define a piecewise linear approximation F of f we need to subdivide $\mathbf{R}^n_+$ into simplices. So, let $\mathcal{G}^n$ be a simplicial subdivision of $\mathbf{R}^n_+$ with some finite mesh. For an appropriate simplicial subdivision of $\mathbf{R}^n_+$ we refer the interested reader to Section 4.

Definition 1 *The piecewise linear approximation F of f with respect to $\mathcal{G}^n$ at a point $x \in \mathbf{R}^n_+$ is given by*

$$F(x) = \sum_{i=1}^{n+1} \lambda_i f(y^i)$$

where the convex hull $\sigma(y^1, \ldots, y^{n+1})$ of $y^1, \ldots, y^{n+1}$ in $\mathbf{R}^n_+$ is an n-dimensional or n-simplex in $\mathcal{G}^n$ containing x and where $\lambda^1, \ldots, \lambda^{n+1} \geq 0$ are such that $x = \sum_{i=1}^{n+1} \lambda_i y^i$ and $\sum_{i=1}^{n+1} \lambda_i = 1$.

The results obtained in Section 2 with respect to f can also be applied to the piecewise linear approximation F of f. In particular, there exists a piecewise linear path $\overline{P}$ of points in $\mathbf{R}^n_+$ starting in v and ending in a solution to (1) with respect to F or going to infinity. For each point x on the path $\overline{P}$ there exists a $\rho \geq 0$

such that for all $i \in \{1, \ldots, n\}$

$$\begin{array}{rcccl} \max\{0,(1-\rho)v_i\} & = & x_i & & \text{if } F_i(x) > 0 \\ \max\{0,(1-\rho)v_i\} & \leq & x_i & \leq & v_i + \rho \quad \text{if } F_i(x) = 0 \\ & & x_i & = & v_i + \rho \quad \text{if } F_i(x) < 0. \end{array} \tag{3}$$

Notice that in (3) the sign pattern of $F(x)$ plays a very important role. Therefore we introduce the notion of a sign vector in $\mathbf{R}^n$.

Definition 2 *A vector $s \in \mathbf{R}^n$ is a sign vector if, for all i, $s_i \in \{-1, 0, +1\}$.*

For a sign vector s in $\mathbf{R}^n$, let $\mathcal{I}^0(s) := \{j \mid s_j = 0\}$, $\mathcal{I}^{+1}(s) := \{j \mid s_j = +1\}$, and $\mathcal{I}^{-1}(s) := \{j \mid s_j = -1\}$. Now, for each sign vector s let the set $\mathcal{A}(s)$ be defined by

$$\mathcal{A}(s) = \emptyset \text{ if } s \geq 0 \text{ and } v_i = 0 \text{ for all } i \in \mathcal{I}^{+1}(s),$$

and otherwise

$$\begin{array}{rllcl} \mathcal{A}(s) = \{x \in \mathbf{R}^n_+ \mid & \text{if } s_i = +1 \text{ then} & \max\{0,(1-\rho)v_i\} & = & x_i, \\ & \text{if } s_i = 0 \text{ then} & \max\{0,(1-\rho)v_i\} & \leq & x_i \leq v_i + \rho, \\ & \text{if } s_i = -1 \text{ then} & & & x_i = v_i + \rho, \\ & \multicolumn{4}{l}{\text{with } \rho \geq 0 \text{ if } s \leq 0 \text{ or } v_i = 0 \text{ for all } i \in \mathcal{I}^{+1}(s),} \\ & \multicolumn{4}{l}{\text{and otherwise } 0 \leq \rho \leq 1\}.} \end{array}$$

For each sign vector s, the set $\mathcal{A}^0(s)$ is defined by

$$\mathcal{A}^0(s) = \emptyset \text{ if } s \leq 0 \text{ or } v_i = 0 \text{ for all } i \in \mathcal{I}^{+1}(s),$$

and otherwise

$$\begin{array}{rllcccl} \mathcal{A}^0(s) = \{x \in \mathbf{R}^n_+ \mid & \text{if } s_i = +1 & \text{then} & 0 & = & x_i, & \\ & \text{if } s_i = 0 & \text{then} & 0 & \leq & x_i & \leq \; v_i + \rho, \\ & \text{if } s_i = -1 & \text{then} & & & x_i & = \; v_i + \rho, \\ & \multicolumn{6}{l}{\text{with } \rho \geq 1\}.} \end{array}$$

Figure 2 gives a subdivision of $\mathbf{R}^n_+$ into subsets $\mathcal{A}$ and $\mathcal{A}^0(s)$ for sign vectors $s \in \mathbf{R}^n$ when $n = 2$. From the definitions of $\mathcal{A}(s)$ and $\mathcal{A}^0(s)$ and from (3), it follows that $x \in \overline{P}$ satisfies $x \in \mathcal{A}(s)$ or $x \in \mathcal{A}^0(s)$ and $s = \text{sgn}(F(x))$, for some sign vector s.

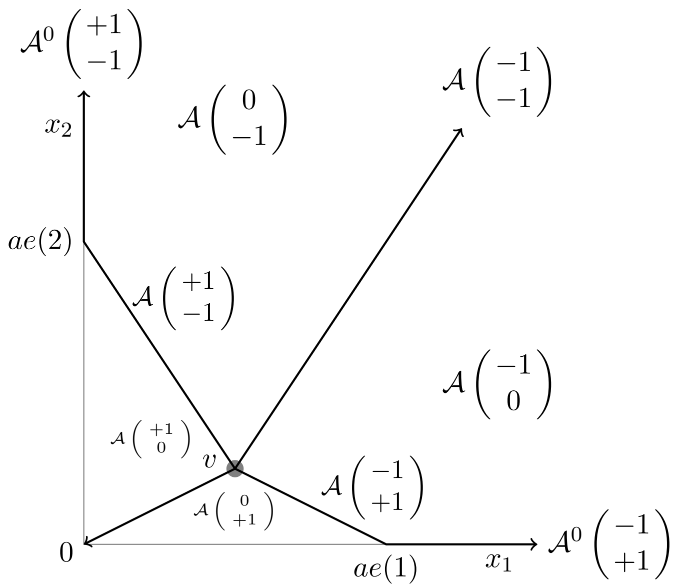

Figure 2: Subdivisions of $\mathbf{R}^2_+$ into subsets $\mathcal{A}$ and $\mathcal{A}^0(s)$ for sign vectors $s \in \mathbf{R}^2$.

The simplicial subdivision $\mathcal{G}^n$ of $\mathbf{R}^n_+$ has to be such that it subdivides each nonempty subset $\mathcal{A}(s)$ and $\mathcal{A}^0(s)$ into t-simplices where t, the dimension of $\mathcal{A}(s)$ ($\mathcal{A}^0(s)$), is equal to $\mid \mathcal{I}^0(s) \mid +1$ (see Section 4 for an appropriate simplicial subdivision). So, if $x \in \mathcal{A}(s)$ (or $x \in \mathcal{A}^0(s)$) there are a t-simplex $\sigma(y^1, \ldots, y^{t+1})$ in $\mathcal{A}(s)$ ($\mathcal{A}^0(s)$) and numbers and numbers $\lambda_1, \ldots, \lambda_{t+1} \geq 0$ such that $x = \sum_{i=1}^{t+1} \lambda_i y^i$ and $\sum_{i=1}^{t+1} \lambda_i = 1$.

On the other hand, if $\text{sgn}(F(x)) = s$, then there exist $\mu_h \geq 0$, $h \notin \mathcal{I}^0(s)$, such that $F(x) = \sum_{h \notin \mathcal{I}^0(s)} \mu_h s_h e(h)$, where $e(h)$ is the n-dimensional unit vector with $e_i(h) = 1$ if $i = h$. Hence, if x lies on the path $\overline{P}$, then for some sign vector s there is a t-simplex $\sigma(y^1, \ldots, y^{t+1})$ in $\mathcal{A}(s)$ ($\mathcal{A}^0(s)$) such that the system of linear equations given by

$$\sum_{i=1}^{t+1} \lambda_i \begin{pmatrix} f(y^i) \\ 1 \end{pmatrix} - \sum_{h \notin \mathcal{I}^0(s)} \mu_h s_h \begin{pmatrix} e(h) \\ 0 \end{pmatrix} = \begin{pmatrix} \underline{0} \\ 1 \end{pmatrix} \tag{4}$$

has a solution $\lambda_i^* \geq 0$, $i = 1, \ldots, t+1$, $\mu_h^* \geq 0$, $h \notin \mathcal{I}^0(s)$, with $x = \sum_{i=1}^{t+1} \lambda_i^* y^i$. The vector $\underline{0}$ in (4) denotes the n-vector of zeroes.

System (4) is a system of $n+1$ equations with $n+2$ unknowns leaving us with a degree of freedom. So, assuming non-degeneracy, a line segment of solutions to (4) exists which can be followed by making a linear programming pivot step in (4). This line segment corresponds to a linear piece of $\overline{P}$ in σ defined by the points $\sum_{i=1}^{t+1} \lambda_i y^i$.

In an end point of a line segment of solutions to (4) either $\lambda_p = 0$ for some $p \in \{1, \ldots, t+1\}$ or $\mu_j = 0$ for some $j \notin \mathcal{I}^0(s)$. If at an end point $\lambda_p = 0$ for some $p \in \{1, \ldots, t+1\}$, then the point $\overline{x} = \sum_{i \neq p} \lambda_i y^i$ lies in the facet τ of σ opposite the vertex y^p. The facet τ is either a facet of exactly one other t-simplex, say $\overline{\sigma}$, in $\mathcal{A}(s)$ ($\mathcal{A}^0(s)$), or it is not and τ lies in the boundary of $\mathcal{A}(s)$ ($\mathcal{A}^0(s)$).

Suppose $\overline{\sigma}$ exists. Then, in order to continue the path $\overline{P}$ in $\mathcal{A}(s)$ ($\mathcal{A}^0(s)$), a pivot step is made in (4) with the column $(f(\overline{y})^T, 1)^T$ corresponding to the unique vertex $\overline{y}$ of $\overline{\sigma}$ not contained in τ. The algorithm is continued by repeating the procedure described.

Suppose $\overline{\sigma}$ does not exist and hence τ lies in the boundary of $\mathcal{A}(s)$ ($\mathcal{A}^0(s)$). If τ lies in $\mathcal{C}^n(s)$, with $\mathcal{C}^n(s)$ defined as $\mathcal{C}^n(s) = \emptyset$ if $s \leq 0$ or $v_i = 0$ for all $i \in \mathcal{I}^{+1}(s)$, and otherwise

$$
\begin{aligned}
\mathcal{C}^n(s) = \{x \in \mathbf{R}^n_+ \mid \quad & \text{if } s_i = +1 \quad \text{then } 0 = x_i, \\
& \text{if } s_i = 0 \qquad \text{then } 0 \leq x_i \leq v_i + 1, \\
& \text{if } s_i = -1 \quad \text{then } \qquad x_i = v_i + 1\},
\end{aligned}
$$

then the algorithm has found a point $\overline{x} \in \mathcal{C}^n(s)$ with sign vector s equal to $sgn(\overline{x})$. If $s \geq 0$ then $\overline{x}$ is an approximate solution for (1). Otherwise τ is the facet of a unique t-simplex σ in $\mathcal{A}^0(s)$ ($\mathcal{A}(s)$). In order to continue the path $\overline{P}$ in $\mathcal{A}^0(s)$ ($\mathcal{A}(s)$), a pivot step is made in (4) with the column $[f(\hat{y})^\top, 1]^\top$ corresponding to the unique vertex $\hat{y}$ of $\hat{\sigma}$ not contained in τ. The algorithm is continued by repeating the procedure described.

If the facet τ of σ in the boundary of $\mathcal{A}(s)$ does not lie in $\mathcal{C}^n(s)$, then τ is a $(t-1)$-simplex in $A(\overline{s})$ or $A^0(\overline{s})$, with $\overline{s}$ being a sign vector such that $\overline{s}_l \neq 0$ for some $l \in \mathcal{I}^0(s)$ while $\overline{s}_i = s_i$ for all $i \neq l$, and the algorithm continues in $A(\overline{s}_l) \neq \emptyset$ for some $l \in \mathcal{I}^0(s)$ by pivoting the column $(\overline{s}_l e(l)^\top, 0)^\top$ into (4).

If at an end point of solutions to (4), μ_j is zero for some $j \notin \mathcal{I}^0(s)$, then at $\overline{s}_j = 0$ and $\overline{s}_h = s_h$ for $h \neq j$. Then $\overline{x} \in \mathcal{A}(s)$ is an approximate solution to (1) if $\overline{s} \geq 0$ and $v_i = 0$ for all $i \in \mathcal{I}^{+1}(\overline{s})$. Also, $\overline{x} \in \mathcal{A}^0(s)$ is an approximate solution to (1) is $\overline{s} \geq 0$. If these conditions do not hold then there is exactly one $(t+1)$-simplex $\overline{\sigma}$ in $A(\overline{s})$ is $\sigma \in \mathcal{A}(s)$ or $\sigma \in \mathcal{A}^0(s)$ and $A^0(\overline{s}) = \emptyset$ ($\overline{\sigma}$ in $A^0(\overline{s})$ if $\sigma \in \mathcal{A}^0(s)$ and $A^0(\overline{s}) \neq \emptyset$)

having σ as a facet. The algorithm continues by pivoting the column $[f(\overline{y}^\top, 1]^\top$ into (4), where $\overline{y}$ is the vertex of $\overline{\sigma}$ not contained in σ.

Now we have described how the algorithm proceeds along the path $\overline{P}$ in the different subsets $\mathcal{A}(s)$ and $\mathcal{A}^0(s)$ of $\mathbf{R}^n_+$, we still have to describe the initialisation of the algorithm at v. At v the system (4) becomes

$$\lambda_1 \begin{pmatrix} f(v) \\ 1 \end{pmatrix} - \sum_{h=1}^{n} s_h^0 \mu_h \begin{pmatrix} e(h) \\ 0 \end{pmatrix} = \begin{pmatrix} \underline{0} \\ 1 \end{pmatrix} \tag{5}$$

having a unique solution $\lambda_1 = 1$, $\mu_h = s_h^0 f_h(v) > 0$, $h \in \{1, \ldots, n\}$, where $s^0 = \text{sgn}(f(v))$. If $s^0 \geq 0$ and $v_i = 0$ for all $i \in \mathcal{I}^{+1}(s^0)$, then the algorithm stops with an exact solution at v. Otherwise, the starting point v is a facet of a unique 1-simplex $\sigma(y^1, y^2)$ in $A(s^0)$ with $y^1 = v$. The algorithm then pivots the column $[f(y^2)^\top, 1]^\top$ into (5).

Since all steps are unique and returning to v is impossible, the algorithm either terminates within a finite number of steps with an approximate solution $\overline{x}$ of (1) or it follows a path towards infinity.

When an approximate solution $\overline{x}$ of (1) is found, one can measure the accuracy of approximation by taking the smallest $\epsilon > 0$ for which for all $i \in \{1, \ldots, n\}$

$$\begin{aligned} -\epsilon \leq f_i(\overline{x}) & \qquad \text{if } x_i = 0 \\ -\epsilon \leq f_i(\overline{x}) \leq \epsilon & \qquad \text{if } x_i > 0. \end{aligned}$$

If $f(\overline{x})$ is not accurate enough, i.e., ϵ is too large, the algorithm is repeated being started at $v = \overline{x}$ with a finer simplicial subdivision of $\mathbf{R}^n_+$. This in the hope to find a more accurate approximation within a finite number of steps.

When a path towards infinity is followed no solution will be found. So, we have to state conditions under which the algorithm converges towards an approximate solution of (1). Theorem (3) states conditions under which an upper bound to the points x generated by the algorithm exists.

Theorem 1 *Let f be a continuous function from $\mathbf{R}^n$ to $\mathbf{R}^n$ and let v be the starting point of the algorithm. Assume there exists a number $\mu > 0$ such that $f_i(x) > 0$ whenever $x_i > \mu$. Then the algorithm terminates within a finite number of steps with an approximate solution of* (1).

Proof: Suppose there exists a number $\mu > 0$ such that $f_i(x) > 0$ whenever $x_i > \mu$. We show that the algorithm cannot generate a t-simplex σ, with vertices $y^1, \ldots, y^{t+1}$, in the simplicial subdivision $\mathcal{G}^n$ of $\mathbf{R}^n_+$ for which for all $x \in \sigma$ holds that $x_i > \mu$ for some $i \in \{1, \ldots, n\}$. Without loss of generality we may assume that $v_i < \mu$. Then, σ lies in $\mathcal{A}(s)$ or $\mathcal{A}^0(s)$, where s is a sign vector such that $s_i \leq 0$, by definition of $\mathcal{A}(s)$ and $\mathcal{A}^0(s)$. If σ is generated by the algorithm we can conclude from (4) that the i-th component of the piecewise linear approximation F in $x = \sum_{j=1}^{t+1} \lambda_j y^j \in \sigma$ is less than or equal to zero, i.e. $F_i(x) \leq 0$. On the other hand, by assumption, $y \in \sigma$ implies $y_i > \mu$, so $f_i(y^j) > 0$ for all vertices y^j, $j = 1, \ldots, t+1$, of σ. Since x is a convex combination of these vertices the piecewise linear approximation F in x, being the same convex combination of $f(y^i)$, $j = 1, \ldots, t+1$, must have a positive i-th component, i.e. $F_i(x) > 0$. This is in contradiction with $F_i(x) \leq 0$. So, σ cannot be generated by the algorithm. $\square$

4 A simplicial subdivision of $\mathbf{R}^n_+$

In order to triangulate $\mathbf{R}^n_+$ one can use any simplicial subdivision. The only restriction one has to pose on the simplicial subdivision of $\mathbf{R}^n_+$ to underly the algorithm described in Section 3 is that it has to triangulate all nonempty subsets $\mathcal{A}(s)$ and $\mathcal{A}^0(s)$. In this section we propose an appropriate simplicial subdivision of $\mathbf{R}^n_+$ which is based on a combination of the V-triangulation as developed in Doup and Talman (1987) and the K'-triangulation as developed in Todd (1976).

To describe the simplicial subdivision of $\mathbf{R}^n_+$ we need to subdivide each nonempty $\mathcal{A}(s)$ into subsets $\mathcal{A}(s, \mathcal{T})$ and each nonempty subset $\mathcal{A}^0(s)$ into subsets $\mathcal{A}^0(s, \mathcal{T})$ with $\mathcal{T} \subset (\mathcal{I}^0(s) \cup -\mathcal{I}^0(s))$ such that for all $i \in \mathcal{I}^0(s)$ either i or $-i$ belongs to $\mathcal{T}$. When $s \not\geq 0$ or $v_i > 0$ for some $i \in \mathcal{I}^{+1}(s)$, then the subsets $\mathcal{A}(s, \mathcal{T})$ are defined as

$$\mathcal{A}(s, \mathcal{T}) = \emptyset \text{ if } v_i = 0 \text{ for some } i \in \mathcal{T},$$

and otherwise

$$\mathcal{A}(s,\mathcal{T}) = \{x \in \mathbf{R}^n_+ \mid \begin{array}{lll} \text{if } s_i = +1 & \text{then} & \max\{0,(1-\rho)v_i\} = \; x_i, \\ \text{if } s_i = 0 \text{ and } i \in \mathcal{T} & \text{then} & \max\{0,(1-\rho)v_i\} \leq \; x_i \leq v_i + \rho, \\ \text{if } s_i = 0 \text{ and } -i \in \mathcal{T} & \text{then} & v_i \leq \; x_i \leq v_i + \rho, \\ \text{if } s_i = -1 & \text{then} & x_i = v_i + \rho, \end{array}$$
$$\text{with } \rho \geq 0 \text{ if } s \leq 0 \text{ or } v_i = 0 \text{ for all } i \in \mathcal{I}^{+1}(s),$$
$$\text{and otherwise } 0 \leq \rho \leq 1\}.$$

When $s \not\leq 0$, $s \not\geq 0$, and $v_i > 0$ for some $i \in \mathcal{I}^{+1}(s)$, then the subsets $\mathcal{A}^0(s,\mathcal{T})$ are defined as

$$\mathcal{A}^0(s) = \emptyset \text{ if } v_i = 0 \text{ for some } i \in \mathcal{T},$$

and otherwise

$$\mathcal{A}^0(s) = \{x \in \mathbf{R}^n_+ \mid \begin{array}{lll} \text{if } s_i = +1 & \text{then} & 0 = x_i \\ \text{if } s_i = 0 \text{ and } i \in \mathcal{T} & \text{then} & 0 \leq x_i \leq v_i \\ \text{if } s_i = 0 \text{ and } -i \in \mathcal{T} & \text{then} & v_i \leq x_i \leq v_i + \rho \\ \text{if } s_i = -1 & \text{then} & x_i = v_i + \rho \end{array}$$
$$\text{with } \rho \geq 1\}.$$

Figure 3 gives a subdivision of $\mathbf{R}^n_+$ into subsets $\mathcal{A}(s,\mathcal{T})$ and $\mathcal{A}^0(s,\mathcal{T})$.

For some positive integer m, each nonempty subset $\mathcal{A}(s,\mathcal{T})$ is subdivided into t-simplices $\sigma(y^1,\pi)$ with vertices $y^1,\ldots,y^{t+1}$, with $t = \mid \mathcal{I}^0(s) \mid +1$ the dimension of $\mathcal{A}(s,\mathcal{T})$, such that

i) $y^1 = v + a(0)m^{-1}q(0) + \sum_{j\in\mathcal{I}^0(s)} a(j)m^{-1}q(j)$ with integers $a(j)$ and $a(0)$ satisfying

  if $s \leq 0$ or $v_i = 0$ for all $i \in \mathcal{I}^{+1}(s)$ then $0 \leq a(j) \leq a(0)$ for all $-j \in \mathcal{T}$, and $\max\{0, a(0)-m\} \leq a(j) \leq a(0)$ for all $j \in \mathcal{T}$, and otherwise $0 \leq a(j) \leq a(0) \leq m-1$ for all $j \in \mathcal{I}^0(s)$;

ii) $\pi = (\pi_1,\ldots,\pi_t)$ is a permutation of the elements of $\mathcal{I}^0(s) \cup \{0\}$ such that for all $j \in \mathcal{I}^0(s)$:

  if $\pi_{p\prime} = 0,\ \pi_p = j$, and $a(\pi_p) = a(\pi_{p\prime})$ then $p\prime < p$;

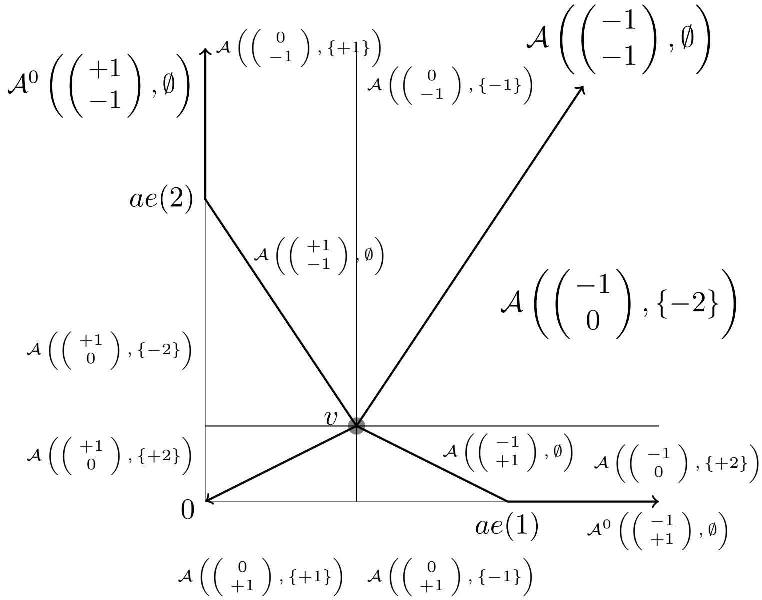

Figure 3: Subdivisions of $\mathbf{R}^2_+$ into subsets $\mathcal{A}(s,\mathcal{T})$ and $\mathcal{A}^0(s,\mathcal{T})$ for sign vectors $s \in \mathbf{R}^2$.

if $\pi_{p\prime} = 0,\ \pi_p = j,\ j \in \mathcal{T}$, and $a(\pi_p) = a(\pi_{p\prime}) - m$ then $p < p\prime$;

iii) $y^{i+1} = y^i + m^{-1}q(\pi_i),\ \ i = 1,\ldots,t;$

where $q_j(0) = 1$ for $j \in \mathcal{I}^{-1}(s)$ or $-j \in \mathcal{T}$ and $q_j(0) = -v_j$ for $j \in \mathcal{I}^{+1}(s)$ or $j \in \mathcal{T}$, and where

$$q(j) = -e(j) \quad \text{if } -j \in \mathcal{T},$$
$$q(j) = v_j e(j) \quad \text{if } j \in \mathcal{T}.$$

Let the simplicial subdivision of $\mathcal{A}(s,\mathcal{T})$ be denoted by $\mathcal{G}^n_m(s,T)$. Then, the simplicial subdivision of $\mathcal{A}(s)$, denoted by $\mathcal{G}^n_m(s)$, is given by the union of $\mathcal{G}^n_m(s,T)$ over all feasible $\mathcal{T}$.

For the same positive integer m, each nonempty subset $\mathcal{A}^0(s,\mathcal{T})$ is subdivided into t-simplices $\sigma(y^1,\pi)$ with vertices $y^1,\ldots,y^{t+1}$, where $t = \mid \mathcal{I}^0(s) \mid +1$ is the dimension of $\mathcal{A}^0(s,\mathcal{T})$, such that

i) $y^1 = v(s,T) + a(0)m^{-1}q(0) + \sum_{j\in\mathcal{I}^0(s)} a(j)m^{-1}q(j)$ with integers $a(j)$ and $a(0)$ satisfying

$a(0) \geq m;$

$0 \leq a(j) \leq a(0)$ for all $-j \in \mathcal{T};$

$a(0) - m \leq a(j) \leq a(0)$ for all $j \in \mathcal{T}$;

$0 \leq a(j) \leq a(0) \leq m - 1$ for all $j \in \mathcal{I}^0(s)$;

ii) $\pi = (\pi_1, \ldots, \pi_t)$ is a permutation of the elements of $\mathcal{I}^0(s) \cup \{0\}$ such that for all $j \in \mathcal{I}^0(s)$:

if $\pi_{p\prime} = 0,\ \pi_p = j$, and $a(\pi_p) = a(\pi_{p\prime})$ then $p\prime < p$;

if $\pi_{p\prime} = 0,\ \pi_p = j,\ j \in \mathcal{T}$, and $a(\pi_p) = a(\pi_{p\prime}) - m$ then $p < p\prime$;

iii) $y^{i+1} = y^i + m^{-1} q(\pi_i),\ \ i = 1, \ldots, t$;

where $v_j(s, T) = 0$ for all $j \in \mathcal{I}^{+1}(s)$ and $v_j(s, T) = v_j$ otherwise, where $q_j(0) = 1$ for $j \in \mathcal{I}^{-1}(s)$ or $-j \in \mathcal{T}$ and $q_j(0) = -v_j$ for $j \in \mathcal{T}$, and $q_j(0) = 0$ otherwise, and where

$$q(j) = -e(j) \quad \text{if } -j \in \mathcal{T},$$
$$q(j) = v_j e(j) \quad \text{if } j \in \mathcal{T}.$$

Let the simplicial subdivision of $\mathcal{A}^0(s, \mathcal{T})$ be denoted by $\mathcal{G}_m^{0n}(s, T)$. Then, the simplicial subdivision of $\mathcal{A}^0(s)$, denoted by $\mathcal{G}_m^{0n}(s)$, is given by the union of $\mathcal{G}_m^{0n}(s, T)$ over all $\mathcal{T}$. The simplicial subdivision of $\mathbf{R}_+^n$, $\mathcal{G}_m^n$, is now induced by the union of $\mathcal{G}_m^n(s)$ and $\mathcal{G}_m^{0n}(s)$ over all possible sign vectors s, m^{-1} being the grid size of the simplicial subdivision. Figure 4 gives a simplicial subdivision of $\mathbf{R}_+^n$ for $n = 2$ and $m = 2$.

In Section 3 we described how to follow the path $\overline{P}$ through $\mathbf{R}_+^n$ from v by making pivot steps in the system of equations (4) with respect to a sequence of adjacent simplices σ in $\mathcal{A}(s)$ or $\mathcal{A}^0(s)$ for varying sign vectors s. After having introduced a specific simplicial subdivision of $\mathbf{R}_+^n$ we now describe how a sequence of adjacent simplices σ in this simplicial subdivision of $\mathbf{R}_+^n$ can be followed, i.e., we describe how, given the parameters y^1, π, and $a(h)$, for $h \in \mathcal{I}^0(s) \cup \{0\}$, of a t-simplex σ, the parameters of a simplex $\overline{\sigma}$ adjacent to σ are obtained.

The movement from a t-simplex $\sigma(y^1, \pi)$ in $\mathcal{A}(s, \mathcal{T})$ ($\mathcal{A}^0(s, \mathcal{T})$) to an adjacent simplex $\overline{\sigma}(\overline{y}^1, \overline{\pi})$ is called a replacement step when $\overline{\sigma}(\overline{y}^1, \overline{\pi})$ is also a t-simplex in $\mathcal{A}(s, \mathcal{T})$ ($\mathcal{A}^0(s, \mathcal{T})$). Making a replacement step we replace a vertex y^p, for some $p \in \{1, \ldots, t+1\}$, of σ opposite the common facet τ of σ and $\overline{\sigma}$ by the vertex $\overline{y}$ of $\overline{\sigma}$ not belonging to τ. The possibilities are listed in Table 1, where the $(n+1)$-vector a is defined by $a_h = a(h)$, $h \in \mathcal{I}^0(s) \cup \{0\}$, and $a_h = 0$ otherwise. Notice that, in Table 1 the unit vector $e(h)$ is an $(n+1) - vector$, $h = 0, 1, \ldots, n$. In case the replacement step with respect to y^p cannot be performed, the

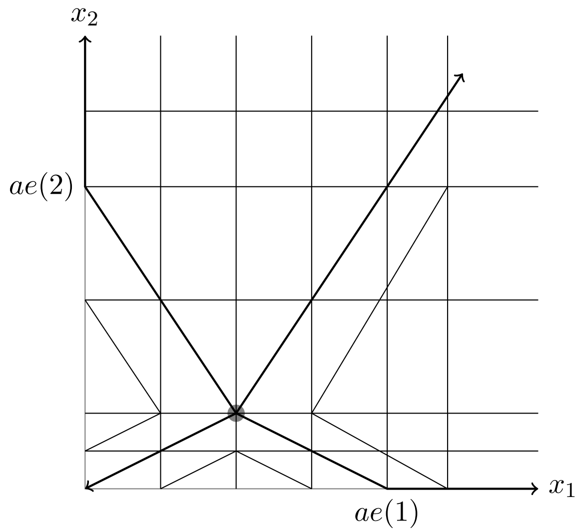

Figure 4: Simplicial subdivisions of $\mathbf{R}^2_+$ with grid size $\frac{1}{2}$.

	$\overline{y}^1$	$\overline{\pi}$	$\overline{a}$
$p = 1$	$y^1 + m^{-1}q(\pi_1)$	$(\pi_2, \ldots, \pi_t, \pi_1)$	$a + e(\pi_1)$
$1 < p < t + 1$	y^1	$(\pi_1, \ldots, \pi_{p-2}, \pi_p, \pi_{p-1}, \pi_{p+1}, \ldots, \pi_t)$	a
$p = t + 1$	$y^1 - m^{-1}q(\pi_t)$	$(\pi_t, \pi_1, \ldots, \pi_{t-1})$	$a - e(\pi_t)$

Table 1: Replacement steps.

facet τ of $\sigma(y^1, \pi)$ opposite y^p lies in the boundary of $\mathcal{A}(s, \mathcal{T})$ ($\mathcal{A}^0(s, \mathcal{T})$). Lemma 1 describes the cases when τ lies in the boundary of $\mathcal{A}(s, \mathcal{T})$ or $\mathcal{A}^0(s, \mathcal{T})$.

Lemma 1 *Let $\sigma(y^1, \pi)$ be a t-simplex in $\mathcal{G}^n_m(s, T)$ ($\mathcal{G}^{0n}_m(s, T)$) and τ the facet of σ opposite vertex y^p, $1 \leq p \leq t + 1$. Then τ lies in the boundary of $\mathcal{A}(s, \mathcal{T})$ ($\mathcal{A}^0(s, \mathcal{T})$) if and only if one of the following properties holds:*

1) $p = 1$, $s \not\leq 0$, $v_i > 0$ *for some* $i \in \mathcal{I}^{+1}(s)$, $\pi_i = 0$, *and* $a(\pi_1) = m - 1$

2) $1 < p < t + 1$, $\pi_{p-1} = k$, $\pi_p = 0$, $k \in \mathcal{T}$, *and* $a(\pi_{p-1}) = a(\pi_p) - m$

3) $1 < p < t + 1$, $\pi_{p-1} = 0$, *and* $a(\pi_{p-1}) = a(\pi_p)$

4) $p = t + 1$, $\pi_t \neq 0$, *and* $a(\pi_t) = 0$

5) $p = t + 1$, $s \not\geq 0$, $s \not\leq 0$, $v_i > 0$ *for some* $i \in \mathcal{I}^{+1}(s)$, $\pi_t = 0$, *and* $a(\pi_t) = m$.

In cases 1 an 5 of Lemma 1, τ lies in $\mathcal{C}^n(s)$. In case 1, if $s \not\geq 0$ then σ lies in $\mathcal{A}(s,\mathcal{T})$ and shares τ with a unique t-simplex $\overline{\sigma}(\overline{y}^1,\overline{\pi})$ in $\mathcal{A}^0(s,\mathcal{T})$ where $\overline{y}^1 = y^1 + m^{-1}q(\pi_t)$ and $\overline{\pi} = (\pi_2,\ldots,\pi_t,\pi_1)$. In case 5, σ lies in $\mathcal{A}^0(s,\mathcal{T})$ and shares τ with a unique t-simplex $\overline{\sigma}(\overline{y}^1,\overline{\pi})$ in $\mathcal{A}(s,\mathcal{T})$ where $\overline{y}^1 = y^1 - m^{-1}q(\pi_t)$ and $\overline{\pi} = (\pi_t,\pi_1,\ldots,\pi_{t-1})$. So, a replacement step is made resulting in a t-simplex in an adjoining set $\mathcal{A}^0(s,\mathcal{T})$ in case 1 ($\mathcal{A}(s,\mathcal{T})$ in case 5). Notice that $q(0)$ is replaced by $\overline{q}(0) = q(0) + \sum_{j\in\mathcal{I}^{+1}(s)} -v_j e(j)$ ($\overline{q}(0) = q(0) - \sum_{j\in\mathcal{I}^{+1}(s)} -v_j e(j)$). In case 2, τ is a $(t-1)$-simplex $\overline{\sigma}(y^1,\overline{\pi})$ in $\mathcal{A}^0(s,\mathcal{T})$ where $\overline{s} = s + e(k)$, $\overline{\mathcal{T}} = \mathcal{T}\setminus\{k\}$, and $\overline{\pi} = (\pi_1,\ldots,\pi_{p-2},\pi_p,\ldots,\pi_t)$. Note that $q(k)$ disappears and $q(0)$ becomes $\overline{q}(0) = q(0) + q(k) - \sum_{j\in\mathcal{I}^{+1}(s)} -v_j e(j)$ when $\sigma\in\mathcal{A}(s,\mathcal{T})$ ($\overline{q}(0) = q(0)+q(k)$ when $\sigma\in\mathcal{A}^0(s,\mathcal{T})$). In case 3, if $\pi_p = k$ and $v_k > 0$, σ shares τ with a t-simplex $\overline{\sigma}(y^1,\pi)$ in $A(s,\overline{\mathcal{T}})$ when $\sigma\in\mathcal{A}(s,\mathcal{T})$ ($A^0(s,\overline{\mathcal{T}})$ when $\sigma\in\mathcal{A}^0(s,\mathcal{T})$) where $\overline{\mathcal{T}} = \mathcal{T}\setminus\{h\}\cup\{-h\}$, $h=-k$ if $-k\in\mathcal{T}$, $h=k$ if $k\in\mathcal{T}$. If in case 3 $\pi_p = k$ and $v_k = 0$ with $-k\in\mathcal{T}$, then τ is a $(t-1)$-simplex $\overline{\sigma}(y^1,\overline{\pi})$ in $A(\overline{s},\overline{\mathcal{T}})$ when $\sigma\in\mathcal{A}(s,\mathcal{T})$ ($A^0(\overline{s},\overline{\mathcal{T}})$ when $\sigma\in\mathcal{A}^0(s,\mathcal{T})$) Here, $q(k)$ disappears and $q(0)$ becomes $\overline{q}(0) = q(0)+q(k)$. In case 4, if $\pi_t = k$ and $-k\in\mathcal{T}$, then τ is a $(t-1)$-simplex $\overline{\sigma}(y^1,\overline{\pi})$ in $A(\overline{s},\overline{\mathcal{T}})$ when $\sigma\in\mathcal{A}(s,\mathcal{T})$ ($A^0(\overline{s},\overline{\mathcal{T}})$ when $\sigma\in\mathcal{A}^0(s,\mathcal{T})$) where $\overline{s} = s - e(k)$, $\overline{\mathcal{T}} = \mathcal{T}\setminus\{-k\}$, and $\overline{\pi} = (\pi_1,\ldots,\pi_{t-1})$. If $\pi_t = k$ and $k\in\mathcal{T}$, then τ is a $(t-1)$-simplex $\overline{\sigma}(y^1,\overline{\pi})$ in $A(\overline{s},\overline{\mathcal{T}})$ where $\overline{s} = s+e(k)$, $\overline{\mathcal{T}} = \mathcal{T}\setminus\{k\}$, and $\overline{\pi} = (\pi_1,\ldots,\pi_{t-1})$. Notice that if $t=1$, $\pi_1 = 0$, and $a(0)=0$ then $\tau = \{v\}$.

Finally, a t-simplex $\sigma(y^1,\pi)$ in $\mathcal{A}(s,\mathcal{T})$ is a facet of exactly one $(t+1)$-simplex $\overline{\sigma}(y^1,\overline{\pi})$ in a nonempty set $A(\overline{s},\overline{\mathcal{T}})$ where $\overline{s}$ is such that $\overline{s}_k = 0$ for a $k\notin\mathcal{I}^0(s)$ while $\overline{s}_i s_i$ for all other $i\in\{1,\ldots,n\}$. If $s_k = +1$ and $v_k \neq 0$ then $\overline{\mathcal{T}} = \mathcal{T}\cup\{k\}$ and $\overline{\pi} = (\pi_1,\ldots,\pi_t,k)$. If $s_k = +1$ and $v_k = 0$ then $\overline{\mathcal{T}} = \mathcal{T}\cup\{-k\}$ and $\overline{\pi} = (\pi_1,\ldots,\pi_{p-1},k,\pi_p,\ldots,\pi_t)$ where p is such that $\pi_p = 0$. If $s_k = -1$ then $\overline{\mathcal{T}} = \mathcal{T}\cup\{-k\}$ and $\overline{\pi} = (\pi_1,\ldots,\pi_t,k)$. A t-simplex $\sigma(y^1,\pi)$ in $\mathcal{A}^0(s,\mathcal{T})$ is a facet of exactly one $(t+1)$-simplex $\overline{\sigma}(y^1,\overline{\pi})$ in a nonempty set $A(\overline{s},\overline{\mathcal{T}})$ if $A^0(\overline{s}) = \emptyset$ and otherwise in $A^0(\overline{s},\overline{\mathcal{T}})$ where $\overline{s}$ is such that $\overline{s}_k = 0$ for all $k\notin\mathcal{I}^0(s)$ while $\overline{s}_i = s_i$ for all other $i\in\{1,\ldots,n\}$. If $s_k = +1$ and $v_k\neq 0$ then $\overline{\mathcal{T}} = \mathcal{T}\cup\{k\}$ and $\overline{\pi} = (\pi_1,\ldots,\pi_{p-1},k,\pi_p,\ldots,\pi_t)$ where p is such that $\pi = 0$. If $s_k = +1$ and $v_k = 0$ then $\overline{\mathcal{T}} = \mathcal{T}\cup\{-k\}$ and $\overline{\pi} = (\pi_1,\ldots,\pi_{p-1},k,\pi_p,\ldots,\pi_t)$ where p is such that $\pi_{p-1} = 0$. If $s_k = -1$ then $\overline{\mathcal{T}} = \mathcal{T}\cup\{-k\}$ and $\overline{\pi} = (\pi_1,\ldots,\pi_t,k)$. This concludes the description of how to follow a sequence of adjacent simplices in a simplicial subdivision.

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